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A decomposition of the natural embedding spaces for the symplectic dual polar spaces

Bart De Bruyn

Ghent University, Department of Pure Mathematics and Computer Algebra, Krijgslaan 281 (S22), B-9000 Gent, Belgium

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Abstract

Let e be the Grassmann-embedding of the symplectic dual polar space $DW(2n-1, \mathbb{K})$ into $\mathrm{PG}(W)$, where W is a $\left[\binom{2n}{n} - \binom{2n}{n-2}\right]$-dimensional vector space over $\mathbb{K}$. For every point z of $DW(2n-1, \mathbb{K})$ and every $i \in \mathbb{N}$, $\Delta_i(z)$ denotes the set of points at distance i from z. We show that for every pair $\{x, y\}$ of mutually opposite points of $DW(2n-1, \mathbb{K})$, W can be written as a direct sum $W_0 \oplus W_1 \oplus \cdots \oplus W_n$ such that the following four properties hold for every $i \in \{0, \ldots, n\}$: (1) $\langle e(\Delta_i(x) \cap \Delta_{n-i}(y))\rangle = \mathrm{PG}(W_i)$; (2) $\langle e(\bigcup_{j\leqslant i} \Delta_j(x))\rangle = \mathrm{PG}(W_0 \oplus W_1 \oplus \cdots \oplus W_i)$; (3) $\langle e(\bigcup_{j\leqslant i} \Delta_j(y))\rangle = \mathrm{PG}(W_{n-i} \oplus W_{n-i+1} \oplus \cdots \oplus W_n)$; (4) $\dim(W_i) = \binom{n}{i}^2 - \binom{n}{i-1} \cdot \binom{n}{i+1}$.

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1. Introduction

With every polar space Π [8],[7, 7.1] of rank $n \geqslant 2$ there is associated a so-called *dual polar space* Δ. This is the point-line geometry whose points and lines are the maximal and next-to-maximal singular subspaces of Π with reverse containment as incidence relation. The polar space

E-mail address: bdb@cage.ugent.be

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Π can easily be retrieved from Δ. Polar spaces of rank 2 are also called *generalized quadrangles* [6]. For a classification of (dual) polar spaces of rank at least 3, we refer to Tits [7].

We will measure distances $\mathrm{d}(\cdot,\cdot)$ between points or non-empty sets of points of Δ in the *collinearity graph* Γ of Δ. The vertices of Γ are the points of Δ and two distinct points are adjacent whenever they are incident with a line. For every point x and every $i \in \mathbb{N}$, let $\Delta_i(x)$, respectively $\Delta_i^*(x)$, denote the set of points at distance i, respectively distance at most i, from x. Dual polar spaces are examples of *near polygons* [5]. These are point-line geometries satisfying the property that for every line L and every point x there exists a unique point on L nearest to x.

Let $\mathrm{PG}(W)$ denote the projective space associated with a vector space W. By a *full projective embedding* of a dual polar space Δ into $\mathrm{PG}(W)$ we mean an injective mapping from the point-set P of Δ to the point-set of $\mathrm{PG}(W)$ satisfying: (E1) the image $e(P)$ of e spans $\mathrm{PG}(W)$; (E2) e maps lines of Δ to lines of $\mathrm{PG}(W)$. The dimension $\dim(W)$ is called the *vector-dimension* of the embedding e.

Let $n \geqslant 2$ and let $V = V(2n, \mathbb{K})$ denote a $2n$-dimensional vector-space over a field $\mathbb{K}$ equipped with a symplectic form $(\cdot,\cdot)$. The subspaces of V which are totally isotropic with respect to $(\cdot,\cdot)$ define a polar space Π. The corresponding dual polar space Δ is denoted by $DW(2n-1,\mathbb{K})$ and is called a *symplectic dual polar space*. Now, let $\bigwedge^n V$ denote the nth exterior power of V. This vector space is $\binom{2n}{n}$-dimensional and the set of all vectors of the form $\bar{e}_{i_1} \wedge \bar{e}_{i_2} \wedge \cdots \wedge \bar{e}_{i_n}$, $1 \leqslant i_1 < i_2 < \cdots < i_n \leqslant 2n$, is a basis of $\bigwedge^n V$. If α is a maximal totally isotropic subspace of V and if $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n$ is a set of vectors generating α, then let $\wedge^n(\alpha)$ denote the 1-dimensional subspace $\langle \bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_n \rangle$ of $\mathrm{PG}(\bigwedge^n V)$. Note that $\wedge^n(\alpha)$ is independent of the generating set $\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n\}$. So, $\wedge^n$ defines a map from the point-set of Δ to the point-set of $\mathrm{PG}(\bigwedge^n V)$. Let W denote the subspace of $\bigwedge^n V$ generated by all 1-spaces $\wedge^n(p)$, where p is a point of $DW(2n-1,\mathbb{K})$. The map $\wedge^n$ then defines a full embedding e of $DW(2n-1,\mathbb{K})$ into $\mathrm{PG}(W)$ which is called the *Grassmann-embedding of* $DW(2n-1,\mathbb{K})$. The vector-dimension $\dim(W)$ of e is equal to $\binom{2n}{n} - \binom{2n}{n-2}$, see Burau [2, 82.7] and Bourbaki [1, 13.3]. The precise value of this dimension will also follow as a by-product from the proof of the main theorem of this paper. This main theorem is as follows:

Theorem 1.1. *Let $n \geqslant 2$, let $\mathbb{K}$ be a field and let V be a $2n$-dimensional vector space over $\mathbb{K}$. Let e denote the Grassmann-embedding of the dual polar space $DW(2n-1,\mathbb{K})$ into the projective space* $\mathrm{PG}(W)$, *where W is some subspace of $\bigwedge^n V$. Then for every pair $\{x, y\}$ of opposite points of $DW(2n-1,\mathbb{K})$ the vector space W can be written as a direct sum $W_0 \oplus W_1 \oplus \cdots \oplus W_n$ such that the following properties hold for every $i \in \{0,\ldots,n\}$*:

(1) $\langle e(\Delta_i(x) \cap \Delta_{n-i}(y))\rangle = \mathrm{PG}(W_i)$;
(2) $\langle e(\Delta_i^*(x))\rangle = \mathrm{PG}(W_0 \oplus W_1 \oplus \cdots \oplus W_i)$;
(3) $\langle e(\Delta_i^*(y))\rangle = \mathrm{PG}(W_{n-i} \oplus W_{n-i+1} \oplus \cdots \oplus W_n)$;
(4) $\dim(W_i) = \binom{n}{i}^2 - \binom{n}{i-1} \cdot \binom{n}{i+1}$.

So, $\dim(W) = \sum_{i=0}^{n} \dim(W_i) = \binom{2n}{n} - \binom{2n}{n-2}$.

Theorem 1.1 plays a crucial rol in Cardinali and De Bruyn [3] for determining more results on the structure of the Grassmann-embedding of the symplectic dual polar space $DW(2n-1,\mathbb{K})$. Theorem 1.1 will be proved in Section 7. Sections 2 till 6 are devoted to develop all the necessary machinery in order to prove this theorem. In the proof, we will use a nice set of $n+1$ convex subspaces of diameter $n-1$ whose images under the Grassmann-embedding generate the whole embedding space, see Sections 4 and 5. This idea is based on Cooperstein [4] where he showed that the subspace of the dual polar space generated by such a set of $n+1$ convex subspaces coincides with the whole point-set if $\mathbb{K}$ is a field of cardinality greater than two.

2. A recursively defined series of numbers

In this section, we define in a recursive way numbers $f_n(k,l)$ ($n \in \mathbb{N} \setminus \{0\}$ and $k,l \in \{0,\ldots,n\}$) and give a closed expression for these numbers.

The numbers $f_1(k,l)$, $k,l \in \{0,1\}$, are defined as follows:

$f_1(k,l)$	$l=0$	$l=1$
$k=0$	1	0
$k=1$	0	1

Now, let $n \geqslant 1$ and $k,l \in \{0,1,\ldots,n+1\}$.

- If k is even and $l=n+1$, then we define $f_{n+1}(k,l) := 0$.
- If k is odd and $l=0$, then we define $f_{n+1}(k,l) := 0$.
- If $k=0$ and $l \neq n+1$, then we define $f_{n+1}(k,l) := \sum_{i=0}^{n} f_n(i,l)$.
- If $k \neq 0$ is even and $l \neq n+1$, then we define $f_{n+1}(k,l) := \sum_{i=k-1}^{n} f_n(i,l)$.
- If k is odd and $l \neq 0$, we define $f_{n+1}(k,l) := \sum_{i=k-1}^{n} f_n(i,l-1)$.

Below, we list the numbers $f_n(k,l)$ for all $n \in \{2,3,4\}$ and all $k,l \in \{0,\ldots,n\}$.

$f_2(k,l)$	$l=0$	$l=1$	$l=2$
$k=0$	1	1	0
$k=1$	0	1	1
$k=2$	0	1	0

$f_3(k,l)$	$l=0$	$l=1$	$l=2$	$l=3$
$k=0$	1	3	1	0
$k=1$	0	1	3	1
$k=2$	0	2	1	0
$k=3$	0	0	1	0

$f_4(k,l)$	$l=0$	$l=1$	$l=2$	$l=3$	$l=4$
$k=0$	1	6	6	1	0
$k=1$	0	1	6	6	1
$k=2$	0	3	5	1	0
$k=3$	0	0	2	2	0
$k=4$	0	0	1	0	0

The aim of this section is to prove that, if $n \geqslant 1$ and $k,l \in \{0,1,\ldots,n\}$, then $f_n(k,l)$ is equal to:

$$g_n(k,l) := \binom{n-1-\lfloor\frac{k}{2}\rfloor}{l-\lfloor\frac{k+1}{2}\rfloor} \cdot \binom{n-\lfloor\frac{k+1}{2}\rfloor}{l+\frac{(-1)^k-1}{2}} - \binom{n-1-\lfloor\frac{k}{2}\rfloor}{l-1-\lfloor\frac{k+1}{2}\rfloor} \cdot \binom{n-\lfloor\frac{k+1}{2}\rfloor}{l+\frac{(-1)^k+1}{2}}.$$

In the above formula, we have followed the following convention: if $r \in \mathbb{N}$ and $s \in \mathbb{Z}$, then $\binom{r}{s} = \frac{r!}{(r-s)!\cdot s!}$ if $0 \leqslant s \leqslant r$ and $\binom{r}{s} = 0$ if either $s < 0$ or $s > r$. Recall also Pascal's identity $\binom{r+1}{s} = \binom{r}{s} + \binom{r}{s-1}$ which holds for every $r \in \mathbb{N}$ and every $s \in \mathbb{Z}$.

Lemma 2.1. *Let $n \in \mathbb{N} \setminus \{0\}$ and let $l \in \{0, \ldots, n\}$. Then $g_n(n, l)$ is equal to 1 if $l = \lfloor \frac{n+1}{2} \rfloor$ and equal to 0 otherwise.*

Proof. We have

$$g_n(n,l) = \binom{n-1-\lfloor \frac{n}{2} \rfloor}{l-\lfloor \frac{n+1}{2} \rfloor} \cdot \binom{n-\lfloor \frac{n+1}{2} \rfloor}{l+\frac{(-1)^n-1}{2}} - \binom{n-1-\lfloor \frac{n}{2} \rfloor}{l-1-\lfloor \frac{n+1}{2} \rfloor} \cdot \binom{n-\lfloor \frac{n+1}{2} \rfloor}{l+\frac{(-1)^n+1}{2}}.$$

If $l \leqslant \lfloor \frac{n+1}{2} \rfloor - 1$, then $g_n(n,l) = 0$, since $l - \lfloor \frac{n+1}{2} \rfloor < 0$ and $l - 1 - \lfloor \frac{n+1}{2} \rfloor < 0$.

If $l = \lfloor \frac{n+1}{2} \rfloor$, then $l - \lfloor \frac{n+1}{2} \rfloor = 0$, $n - \lfloor \frac{n+1}{2} \rfloor = l + \frac{(-1)^n-1}{2}$ and $l - 1 - \lfloor \frac{n+1}{2} \rfloor < 0$. Hence, $g_n(n,l) = 1$.

If $l \geqslant \lfloor \frac{n+1}{2} \rfloor + 1$, then $g_n(n,l) = 0$, since $n - \lfloor \frac{n+1}{2} \rfloor < l + \frac{(-1)^n-1}{2}$ and $n - \lfloor \frac{n+1}{2} \rfloor < l + \frac{(-1)^n+1}{2}$. □

Lemma 2.2. *Let $n \in \mathbb{N} \setminus \{0\}$ and $l \in \{0, \ldots, n\}$. Then $f_n(n, l)$ is equal to 1 if $l = \lfloor \frac{n+1}{2} \rfloor$ and equal to 0 otherwise.*

Proof. We will prove the lemma by induction on n. Obviously, the lemma holds if $n = 1$. So, suppose $n \geqslant 2$ and that the lemma holds for smaller values of n.

Suppose first that n is even. If $l = n$, then $f_n(n,l) = 0$ and $l \neq \lfloor \frac{n+1}{2} \rfloor$. If $l \neq n$, then

$$\begin{aligned} f_n(n,l) &= \sum_{i=n-1}^{n-1} f_{n-1}(i,l) \\ &= f_{n-1}(n-1,l). \end{aligned}$$

By the induction hypothesis, $f_{n-1}(n-1,l)$ is equal to 1 if $l = \lfloor \frac{n}{2} \rfloor = \lfloor \frac{n+1}{2} \rfloor$ and equal to 0 otherwise.

Suppose next that n is odd. If $l = 0$, then $f_n(n,l) = 0$ and $0 \neq \lfloor \frac{n+1}{2} \rfloor$. If $l \neq 0$, then

$$\begin{aligned} f_n(n,l) &= \sum_{i=n-1}^{n-1} f_{n-1}(i,l-1) \\ &= f_{n-1}(n-1,l-1). \end{aligned}$$

By the induction hypothesis, $f_{n-1}(n-1,l-1)$ is equal to 1 if $l = \lfloor \frac{n}{2} \rfloor + 1 = \lfloor \frac{n+1}{2} \rfloor$ and equal to 0 otherwise. This proves the lemma. □

Proposition 2.3. *For all $n \geqslant 1$ and $k, l \in \{0, \ldots, n\}$, $f_n(k,l) = g_n(k,l)$.*

Proof. It is straightforward to verify that $g_n(k,l) = 0$ if (k even and $l = n$) or (k odd and $l = 0$). So, the proposition certainly holds in these cases.

We will prove the proposition by induction on the pairs (n, k) with $n \in \mathbb{N} \setminus \{0\}$ and $k \in \{0, \ldots, n\}$. We will use the following ordering: for two such pairs (n_1, k_1) and (n_2, k_2), we say that $(n_1, k_1) < (n_2, k_2)$ if either $(n_1 < n_2)$ or $(n_1 = n_2$ and $k_1 > k_2)$.

Obviously, the proposition holds for all pairs (n, k) with $n = 1$. So, suppose $n \geqslant 2$ and that the proposition holds for all pairs (n', k') with $(n', k') < (n, k)$.

- If $k = n$, then the claim follows from Lemmas 2.1 and 2.2.
- If $k = 0$ and $l \neq n$, then $f_n(0, l) = \sum_{i=0}^{n-1} f_{n-1}(i, l) = f_n(1, l + 1)$. By the induction hypothesis, $f_n(1, l + 1) = g_n(1, l + 1) = \binom{n-1}{l} \cdot \binom{n-1}{l} - \binom{n-1}{l-1} \cdot \binom{n-1}{l+1} = \binom{n-1}{l} \cdot \left[\binom{n-1}{l} + \binom{n-1}{l-1}\right] - \binom{n-1}{l-1} \cdot \left[\binom{n-1}{l+1} + \binom{n-1}{l}\right] = \binom{n-1}{l} \cdot \binom{n}{l} - \binom{n-1}{l-1} \cdot \binom{n}{l+1} = g_n(0, l)$. Hence, $f_n(0, l) = g_n(0, l)$.

In the sequel, we suppose that $k \in \{1, \ldots, n-1\}$ with $l \neq n$ if k is even and $l \neq 0$ if k is odd. Suppose first that k is even. Then

$$\begin{aligned} f_n(k, l) &= \sum_{i=k-1}^{n-1} f_{n-1}(i, l) \\ &= f_{n-1}(k-1, l) + \sum_{i=k}^{n-1} f_{n-1}(i, l) \\ &= f_{n-1}(k-1, l) + f_n(k+1, l+1). \end{aligned}$$

By the induction hypothesis,

$$f_{n-1}(k-1, l) = \binom{n - \frac{k}{2} - 1}{l - \frac{k}{2}} \cdot \binom{n - \frac{k}{2} - 1}{l - 1} - \binom{n - \frac{k}{2} - 1}{l - \frac{k}{2} - 1} \cdot \binom{n - \frac{k}{2} - 1}{l},$$

$$f_n(k+1, l+1) = \binom{n - \frac{k}{2} - 1}{l - \frac{k}{2}} \cdot \binom{n - \frac{k}{2} - 1}{l} - \binom{n - \frac{k}{2} - 1}{l - \frac{k}{2} - 1} \cdot \binom{n - \frac{k}{2} - 1}{l + 1},$$

and hence

$$\begin{aligned} f_n(k, l) &= \binom{n - \frac{k}{2} - 1}{l - \frac{k}{2}} \cdot \binom{n - \frac{k}{2}}{l} - \binom{n - \frac{k}{2} - 1}{l - \frac{k}{2} - 1} \cdot \binom{n - \frac{k}{2}}{l + 1} \\ &= g_n(k, l). \end{aligned}$$

Suppose next that k is odd. Then

$$\begin{aligned} f_n(k, l) &= \sum_{i=k-1}^{n-1} f_{n-1}(i, l-1) \\ &= f_{n-1}(k-1, l-1) + \sum_{i=k}^{n-1} f_{n-1}(i, l-1) \\ &= f_{n-1}(k-1, l-1) + f_n(k+1, l-1). \end{aligned}$$

By the induction hypothesis,

$$f_{n-1}(k-1,l-1) = \binom{n-\frac{k+3}{2}}{l-\frac{k+1}{2}} \cdot \binom{n-\frac{k+1}{2}}{l-1} - \binom{n-\frac{k+3}{2}}{l-\frac{k+3}{2}} \cdot \binom{n-\frac{k+1}{2}}{l},$$

$$f_n(k+1,l-1) = \binom{n-\frac{k+3}{2}}{l-\frac{k+3}{2}} \cdot \binom{n-\frac{k+1}{2}}{l-1} - \binom{n-\frac{k+3}{2}}{l-\frac{k+5}{2}} \cdot \binom{n-\frac{k+1}{2}}{l},$$

and hence

$$\begin{aligned} f_n(k,l) &= \binom{n-\frac{k+1}{2}}{l-\frac{k+1}{2}} \cdot \binom{n-\frac{k+1}{2}}{l-1} - \binom{n-\frac{k+1}{2}}{l-\frac{k+3}{2}} \cdot \binom{n-\frac{k+1}{2}}{l} \\ &= g_n(k,l). \qquad \square \end{aligned}$$

3. Special bases in symplectic vector spaces

Let V be a $2n$-dimensional vector space ($n \geqslant 1$) over a field $\mathbb{K}$, and let $(\cdot,\cdot)$ be a non-degenerate symplectic form in V.

A k-tuple $(\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_k)$ of vectors of V is called *nice* if $\langle \bar{g}_1, \bar{g}_2, \ldots, \bar{g}_k \rangle$ is a totally isotropic subspace of dimension k, i.e. if $\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_k$ are linearly independent and if $(\bar{g}_i, \bar{g}_j) = 0$ for all $i, j \in \{1, 2, \ldots, k\}$. We denote by Ω_k the set of all k-tuples $(\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_k)$ of vectors of V satisfying:

(i) the vectors $\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_k$ are linearly independent;
(ii) $(\bar{g}_i, \bar{g}_j) = 0$ for all $i, j \in \{1, \ldots, k\}$ with $|i-j| \neq 1$ and $(\bar{g}_i, \bar{g}_j) = 1$ for all $i, j \in \{1, \ldots, k\}$ with $j - i = 1$.

Lemma 3.1. *Let $(\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_k) \in \Omega_k$, $k \geqslant 1$. Then $\langle \bar{g}_1, \bar{g}_2, \ldots, \bar{g}_k \rangle$ is a non-degenerate subspace if and only if k is even. If k is odd, then the radical of $\langle \bar{g}_1, \bar{g}_2, \ldots, \bar{g}_k \rangle$ is given by $\langle \bar{g}_1 + \bar{g}_3 + \cdots + \bar{g}_k \rangle$.*

Proof. Let $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{K}$ such that

$$(\bar{g}_i, \alpha_1\bar{g}_1 + \alpha_2\bar{g}_2 + \cdots + \alpha_k\bar{g}_k) = 0$$

for every $i \in \{1, \ldots, k\}$. Then $\alpha_2 = 0$, $\alpha_{k-1} = 0$ and $\alpha_i = \alpha_{i+2}$ for all $i \in \{1, \ldots, k-2\}$. If k is even, it follows that $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$. If k is odd, then $\alpha_2 = \alpha_4 = \cdots = \alpha_{k-1} = 0$ and $\alpha_1 = \alpha_3 = \cdots = \alpha_k$. The lemma follows. $\square$

An element of Ω_{2n} can be regarded as an ordered basis of V. We call the elements of Ω_{2n} *special bases* of V. Special bases do exist by the following lemma.

Lemma 3.2. *Let $k \in \{1, \ldots, 2n-1\}$. Then every element $(\bar{g}_1, \ldots, \bar{g}_k)$ of Ω_k can be completed to an element $(\bar{g}_1, \ldots, \bar{g}_k, \bar{g}_{k+1}, \ldots, \bar{g}_{2n})$ of Ω_{2n}.*

Proof. We define the points $\bar{g}_i$, $i \in \{k+1, \ldots, 2n\}$, in a recursive way. For every $i \in \{k+1, \ldots, 2n\}$, let $\bar{g}_i$ be a vector of $\langle \bar{g}_1, \ldots, \bar{g}_{i-2} \rangle^\perp \setminus (\langle \bar{g}_1, \ldots, \bar{g}_{i-1} \rangle^\perp \cup \langle \bar{g}_1, \ldots, \bar{g}_{i-1} \rangle)$ such that $(\bar{g}_{i-1}, \bar{g}_i) = 1$. Such a vector exists for the following reasons:

(i) $\langle \bar{g}_1, \ldots, \bar{g}_{i-1}\rangle^\perp$ is a hyperplane of $\langle \bar{g}_1, \ldots, \bar{g}_{i-2}\rangle^\perp$ and the dimension of $\langle \bar{g}_1, \ldots, \bar{g}_{i-2}\rangle^\perp$ is at least 2;

(ii) $\langle \bar{g}_1, \ldots, \bar{g}_{i-2}\rangle^\perp \cap \langle \bar{g}_1, \ldots, \bar{g}_{i-1}\rangle$ is a 1-dimensional subspace which is equal to $\langle \bar{g}_1 + \bar{g}_3 + \cdots + \bar{g}_{i-2}\rangle$ if i is odd and $\langle \bar{g}_1 + \bar{g}_3 + \cdots + \bar{g}_{i-1}\rangle$ if i is even. □

Remark. If $k \geqslant 2$ is even and if $(\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_k) \in \Omega_k$, then $(\cdot, \cdot)$ induces a non-degenerate symplectic form on the subspace $\langle \bar{g}_1, \bar{g}_2, \ldots, \bar{g}_k\rangle$ and $(\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_k)$ is a special basis of this symplectic vector space.

4. Some subspaces of $\bigwedge^n V$

We continue with the notations introduced in Section 3. Let W denote the subspace of $\bigwedge^n V$ generated by all vectors $\bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_n$, where $(\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n)$ is a nice n-tuple of vectors of V. For every nonzero vector $\bar{g}$ of V, let $W(\bar{g})$ denote the subspace of W generated by all vectors $\bar{g} \wedge \bar{v}_2 \wedge \bar{v}_3 \wedge \cdots \wedge \bar{v}_n$, where $(\bar{g}, \bar{v}_2, \bar{v}_3, \ldots, \bar{v}_n)$ is a nice n-tuple.

Remark. The following holds for all vectors $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n \in V$, for all $i \in \{1, \ldots, n\}$ and all $a_j \in \mathbb{K}$ with $j \in \{1, \ldots, n\} \setminus \{i\}$:

$$\bar{v}_1 \wedge \cdots \wedge \bar{v}_i \wedge \cdots \wedge \bar{v}_n = \bar{v}_1 \wedge \cdots \wedge \bar{v}_{i-1} \wedge \left(\bar{v}_i + \sum_{j \neq i} a_j \bar{v}_j\right) \wedge \bar{v}_{i+1} \wedge \cdots \wedge \bar{v}_n, \tag{1}$$

$$\bar{v}_1 \wedge \cdots \wedge \bar{v}_i \wedge \bar{v}_{i+1} \wedge \cdots \wedge \bar{v}_n = \bar{v}_1 \wedge \cdots \wedge \bar{v}_{i+1} \wedge (-\bar{v}_i) \wedge \cdots \wedge \bar{v}_n \quad (i \neq n). \tag{2}$$

Lemma 4.1. *If $\bar{h}_1$ and $\bar{h}_2$ are two non-orthogonal vectors, then $W(\lambda_1\bar{h}_1 + \lambda_2\bar{h}_2) \subseteq \langle W(\bar{h}_1), W(\bar{h}_2)\rangle$ for all $\lambda_1, \lambda_2 \in \mathbb{K}$ with $(\lambda_1, \lambda_2) \neq (0, 0)$.*

Proof. Obviously, this holds if $n = 1$. So, suppose $n \geqslant 2$.

Let $(\lambda_1\bar{h}_1 + \lambda_2\bar{h}_2) \wedge \bar{v}_2 \wedge \bar{v}_3 \wedge \cdots \wedge \bar{v}_n$ be a vector of $W(\lambda_1\bar{h}_1 + \lambda_2\bar{h}_2)$, where $(\lambda_1\bar{h}_1 + \lambda_2\bar{h}_2, \bar{v}_2, \bar{v}_3, \ldots, \bar{v}_n)$ is a nice n-tuple. Without loss of generality, we may suppose that $\bar{v}_2, \ldots, \bar{v}_n$ are orthogonal with $\bar{h}_1$ and $\bar{h}_2$. (Otherwise, consider for each $i \in \{2, \ldots, n\}$ a suitable linear combination of $\bar{v}_i$ and $\lambda_1\bar{h}_1 + \lambda_2\bar{h}_2$ and apply equation (1)). Clearly,

$$(\lambda_1\bar{h}_1 + \lambda_2\bar{h}_2) \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_n = \lambda_1(\bar{h}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_n) + \lambda_2(\bar{h}_2 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_n)$$

belongs to $\langle W(\bar{h}_1), W(\bar{h}_2)\rangle$. Since $W(\lambda_1\bar{h}_1 + \lambda_2\bar{h}_2)$ is generated by vectors of the form $(\lambda_1\bar{h}_1 + \lambda_2\bar{h}_2) \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_n$, $W(\lambda_1\bar{h}_1 + \lambda_2\bar{h}_2) \subseteq \langle W(\bar{h}_1), W(\bar{h}_2)\rangle$. □

Lemma 4.2. *Let $(\bar{g}_1, \bar{g}_2, \bar{g}_3) \in \Omega_3$. Then for all $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{K}$ with $(\lambda_1, \lambda_2, \lambda_3) \neq (0, 0, 0)$, $W(\lambda_1\bar{g}_1 + \lambda_2\bar{g}_2 + \lambda_3\bar{g}_3) \subseteq \langle W(\bar{g}_1), W(\bar{g}_2), W(\bar{g}_3)\rangle$.*

Proof. By Lemma 3.1, the subspace $\langle \bar{g}_1, \bar{g}_2, \bar{g}_3\rangle$ is degenerate and its radical is given by $\langle \bar{g}_1 + \bar{g}_3\rangle$. If $\bar{g}$ is a vector of $\langle \bar{g}_1, \bar{g}_2, \bar{g}_3\rangle \setminus \langle \bar{g}_1 + \bar{g}_3\rangle$, then by (successive application of) Lemma 4.1, $W(\bar{g}) \subseteq \langle W(\bar{g}_1), W(\bar{g}_2), W(\bar{g}_3)\rangle$. It remains to show that also $W(\bar{g}_1 + \bar{g}_3) \subseteq \langle W(\bar{g}_1), W(\bar{g}_2), W(\bar{g}_3)\rangle$.

Notice that by Lemma 3.2, $(\bar{g}_1, \bar{g}_2, \bar{g}_3)$ can be completed to a special basis $(\bar{g}_1, \bar{g}_2, \bar{g}_3, \dots, \bar{g}_{2n})$ of V.

Consider first the case $n = 2$. Then $W(\bar{g}_1 + \bar{g}_3)$ is generated by the vectors $(\bar{g}_1 + \bar{g}_3) \wedge \bar{g}_1 \in W(\bar{g}_1)$ and $(\bar{g}_1 + \bar{g}_3) \wedge \bar{g}_2 \in W(\bar{g}_2)$. Hence, $W(\bar{g}_1 + \bar{g}_3) \subseteq \langle W(\bar{g}_1), W(\bar{g}_2), W(\bar{g}_3)\rangle$.

Consider next the case $n = 3$. An element of $W(\bar{g}_1)$ is of the form $\bar{g}_1 \wedge (a_1\bar{g}_3 + a_2\bar{g}_4 + a_3\bar{g}_5 + a_4\bar{g}_6) \wedge (b_1\bar{g}_3 + b_2\bar{g}_4 + b_3\bar{g}_5 + b_4\bar{g}_6)$, where $a_1b_2 + a_2b_3 + a_3b_4 - a_2b_1 - a_3b_2 - a_4b_3 = 0$. One readily verifies that $W(\bar{g}_1)$ is generated by the vectors

$$\bar{g}_1 \wedge \bar{g}_3 \wedge \bar{g}_5, \quad \bar{g}_1 \wedge \bar{g}_3 \wedge \bar{g}_6, \quad \bar{g}_1 \wedge \bar{g}_4 \wedge (\bar{g}_3 + \bar{g}_5),$$
$$\bar{g}_1 \wedge \bar{g}_4 \wedge \bar{g}_6, \quad \bar{g}_1 \wedge \bar{g}_5 \wedge (\bar{g}_4 + \bar{g}_6).$$

In a similar way, one can verify that the space $W(\bar{g}_2)$ is generated by the vectors

$$\bar{g}_2 \wedge (\bar{g}_1 + \bar{g}_3) \wedge \bar{g}_5, \quad \bar{g}_2 \wedge (\bar{g}_1 + \bar{g}_3) \wedge \bar{g}_6, \quad \bar{g}_2 \wedge \bar{g}_4 \wedge (\bar{g}_1 + \bar{g}_3 + \bar{g}_5),$$
$$\bar{g}_2 \wedge \bar{g}_4 \wedge \bar{g}_6, \quad \bar{g}_2 \wedge \bar{g}_5 \wedge (\bar{g}_4 + \bar{g}_6).$$

The space $W(\bar{g}_3)$ is generated by the vectors

$$\bar{g}_3 \wedge \bar{g}_1 \wedge \bar{g}_5, \quad \bar{g}_3 \wedge \bar{g}_1 \wedge \bar{g}_6, \quad \bar{g}_3 \wedge (\bar{g}_2 + \bar{g}_4) \wedge \bar{g}_6,$$
$$\bar{g}_3 \wedge (\bar{g}_2 + \bar{g}_4) \wedge (\bar{g}_1 + \bar{g}_5), \quad \bar{g}_3 \wedge \bar{g}_5 \wedge (\bar{g}_2 + \bar{g}_4 + \bar{g}_6).$$

The space $W(\bar{g}_1 + \bar{g}_3)$ is generated by the vectors

$$(\bar{g}_1 + \bar{g}_3) \wedge \bar{g}_1 \wedge \bar{g}_5, \quad (\bar{g}_1 + \bar{g}_3) \wedge \bar{g}_1 \wedge \bar{g}_6, \quad (\bar{g}_1 + \bar{g}_3) \wedge \bar{g}_2 \wedge \bar{g}_5,$$
$$(\bar{g}_1 + \bar{g}_3) \wedge \bar{g}_2 \wedge \bar{g}_6, \quad (\bar{g}_1 + \bar{g}_3) \wedge (\bar{g}_1 + \bar{g}_6) \wedge (\bar{g}_2 + \bar{g}_5).$$

Obviously, $(\bar{g}_1 + \bar{g}_3) \wedge \bar{g}_1 \wedge \bar{g}_5, (\bar{g}_1 + \bar{g}_3) \wedge \bar{g}_1 \wedge \bar{g}_6 \in W(\bar{g}_1)$ and $(\bar{g}_1 + \bar{g}_3) \wedge \bar{g}_2 \wedge \bar{g}_5, (\bar{g}_1 + \bar{g}_3) \wedge \bar{g}_2 \wedge \bar{g}_6 \in W(\bar{g}_2)$. The vector $(\bar{g}_1 + \bar{g}_3) \wedge (\bar{g}_1 + \bar{g}_6) \wedge (\bar{g}_2 + \bar{g}_5)$ is equal to

$$\bar{g}_1 \wedge \bar{g}_6 \wedge \bar{g}_2 + \bar{g}_1 \wedge \bar{g}_6 \wedge \bar{g}_5 + \bar{g}_3 \wedge \bar{g}_1 \wedge \bar{g}_2 + \bar{g}_3 \wedge \bar{g}_1 \wedge \bar{g}_5 + \bar{g}_3 \wedge \bar{g}_6 \wedge \bar{g}_2 + \bar{g}_3 \wedge \bar{g}_6 \wedge \bar{g}_5$$

and can be written as

$$(\bar{g}_2 \wedge (\bar{g}_1 + \bar{g}_3) \wedge \bar{g}_6) - (\bar{g}_1 \wedge \bar{g}_3 \wedge \bar{g}_5) - (\bar{g}_1 \wedge \bar{g}_5 \wedge (\bar{g}_4 + \bar{g}_6))$$
$$-(\bar{g}_3 \wedge (\bar{g}_2 + \bar{g}_4) \wedge (\bar{g}_1 + \bar{g}_5)) - (\bar{g}_1 \wedge \bar{g}_4 \wedge (\bar{g}_3 + \bar{g}_5)) - (\bar{g}_3 \wedge \bar{g}_5 \wedge (\bar{g}_2 + \bar{g}_4 + \bar{g}_6)).$$

It follows that $W(\bar{g}_1 + \bar{g}_3) \subseteq \langle W(\bar{g}_1), W(\bar{g}_2), W(\bar{g}_3)\rangle$.

Finally consider the case $n \geqslant 4$. The space $W(\bar{g}_1 + \bar{g}_3)$ is generated by the vectors $(\bar{g}_1 + \bar{g}_3) \wedge \bar{v}_2 \wedge \bar{v}_3 \wedge \cdots \wedge \bar{v}_n$, where $(\bar{g}_1 + \bar{g}_3, \bar{v}_2, \bar{v}_3, \dots, \bar{v}_n)$ is a nice n-tuple. By Eqs. (1) and (2), we may suppose that the vectors $\bar{v}_4, \dots, \bar{v}_n$ are orthogonal with $\bar{g}_1$, $\bar{g}_2$ and $\bar{g}_3$. Now, consider the quotient space $\langle \bar{v}_4, \dots, \bar{v}_n\rangle^{\perp}/\langle \bar{v}_4, \dots, \bar{v}_n\rangle$ with the symplectic form inherited from V. This is a 6-dimensional symplectic space over $\mathbb{K}$. Applying the reasoning of the case $n = 3$ to the vectors $\bar{g}_1 + \langle \bar{v}_4, \dots, \bar{v}_n\rangle$, $\bar{g}_2 + \langle \bar{v}_4, \dots, \bar{v}_n\rangle$, $\bar{g}_3 + \langle \bar{v}_4, \dots, \bar{v}_n\rangle$ and $\bar{g}_1 + \bar{g}_3 + \langle \bar{v}_4, \dots, \bar{v}_n\rangle$, we see that $(\bar{g}_1 + \bar{g}_3) \wedge \bar{v}_2 \wedge \bar{v}_3 \wedge \cdots \wedge \bar{v}_n \in \langle W(\bar{g}_1), W(\bar{g}_2), W(\bar{g}_3)\rangle$. Hence, $W(\bar{g}_1 + \bar{g}_3) \subseteq \langle W(\bar{g}_1), W(\bar{g}_2), W(\bar{g}_3)\rangle$. $\square$

Lemma 4.3. *Let $(\bar{g}_1, \bar{g}_2, \dots, \bar{g}_k) \in \Omega_k$, $k \geqslant 1$. Then for every nonzero vector $\bar{g} \in \langle \bar{g}_1, \bar{g}_2, \dots, \bar{g}_k\rangle$, $W(\bar{g}) \subseteq \langle W(\bar{g}_1), W(\bar{g}_2), \dots, W(\bar{g}_k)\rangle$.*

Proof. We will prove this by induction on k. The claim holds if $k \leqslant 3$ by Lemmas 4.1 and 4.2. So, we will suppose that $k \geqslant 4$ and that the lemma holds for elements of $\Omega_{k'}$ with $1 \leqslant k' \leqslant k-1$.

(i) Let $\bar{g}$ be a vector of $\langle \bar{g}_1, \bar{g}_2, \ldots, \bar{g}_k \rangle$ not contained in $\bar{g}_k^{\perp}$. Then the 2-space $\langle \bar{g}, \bar{g}_k \rangle$ intersects $\langle \bar{g}_1, \ldots, \bar{g}_{k-1} \rangle$ in a 1-space $\langle \bar{g}' \rangle$. Since $(\bar{g}_k, \bar{g}') \neq 0$, $W(\bar{g}) \subseteq \langle W(\bar{g}_k), W(\bar{g}') \rangle$ by Lemma 4.1. By the induction hypothesis, we know that $W(\bar{g}') \subseteq \langle W(\bar{g}_1), \ldots, W(\bar{g}_{k-1}) \rangle$. Hence, $W(\bar{g}) \subseteq \langle W(\bar{g}_1), \ldots, W(\bar{g}_k) \rangle$.

(ii) Let $\bar{g}$ be a vector of $\langle \bar{g}_1, \bar{g}_2, \ldots, \bar{g}_k \rangle \cap \bar{g}_k^{\perp}$ not contained in the radical of $\langle \bar{g}_1, \bar{g}_2, \ldots, \bar{g}_k \rangle$. Let α be a two-dimensional subspace of $\langle \bar{g}_1, \bar{g}_2, \ldots, \bar{g}_k \rangle$ through $\bar{g}$ not contained in $(\bar{g}_k^{\perp} \cap \langle \bar{g}_1, \bar{g}_2, \ldots, \bar{g}_k \rangle) \cup (\bar{g}^{\perp} \cap \langle \bar{g}_1, \bar{g}_2, \ldots, \bar{g}_k \rangle)$. Let $\bar{g}'$ be an arbitrary vector of $\alpha \setminus \langle \bar{g} \rangle$. Then $(\bar{g}', \bar{g} + \bar{g}') \neq 0$ and hence $W(\bar{g}) \subseteq \langle W(\bar{g}'), W(\bar{g} + \bar{g}') \rangle$ by Lemma 4.1. Since $\bar{g}' \notin \bar{g}_k^{\perp}$ and $\bar{g} + \bar{g}' \notin \bar{g}_k^{\perp}$, $W(\bar{g}')$ and $W(\bar{g} + \bar{g}')$ are contained in $\langle W(\bar{g}_1), \ldots, W(\bar{g}_k) \rangle$. It follows that also $W(\bar{g})$ is contained in $\langle W(\bar{g}_1), \ldots, W(\bar{g}_k) \rangle$.

If k is even, then the lemma holds by (i), (ii) and Lemma 3.1. If k is odd, then we still need to show that the lemma holds for a nonzero vector $\bar{g}^*$ in the radical of $\langle \bar{g}_1, \bar{g}_2, \ldots, \bar{g}_k \rangle$. Obviously, $(\bar{g}_1, \bar{g}_2, \bar{g}^* - \bar{g}_1) \in \Omega_3$. By Lemma 4.2, we have $W(\bar{g}^*) \subseteq \langle W(\bar{g}_1), W(\bar{g}_2), W(\bar{g}^* - \bar{g}_1) \rangle$. By (i) and (ii), $W(\bar{g}^* - \bar{g}_1) \subseteq \langle W(\bar{g}_1), \ldots, W(\bar{g}_k) \rangle$. Hence, $W(\bar{g}^*) \subseteq \langle W(\bar{g}_1), W(\bar{g}_2), \ldots, W(\bar{g}_k) \rangle$. $\square$

Proposition 4.4. *If* $(\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_{n+1}) \in \Omega_{n+1}$, *then* $W = \langle W(\bar{g}_1), W(\bar{g}_2), \ldots, W(\bar{g}_{n+1}) \rangle$.

Proof. It suffices to show that for every nice n-tuple $(\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n)$, $\bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_n \in \langle W(\bar{g}_1), \ldots, W(\bar{g}_{n+1}) \rangle$. The n-dimensional subspace $\langle \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n \rangle$ of V intersects $\langle \bar{g}_1, \bar{g}_2, \ldots, \bar{g}_{n+1} \rangle$ in a subspace of dimension at least 1. Let $\bar{w}$ denote a nonzero vector in this intersection. Then there exists a nice n-tuple $(\bar{w}, \bar{v}_2', \ldots, \bar{v}_n')$ such that $\bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_n = \bar{w} \wedge \bar{v}_2' \wedge \cdots \wedge \bar{v}_n'$. Hence, $\bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_n \in W(\bar{w})$. By Lemma 4.3, we know that $W(\bar{w}) \subseteq \langle W(\bar{g}_1), W(\bar{g}_2), \ldots, W(\bar{g}_{n+1}) \rangle$. Hence, $\bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_n \in \langle W(\bar{g}_1), W(\bar{g}_2), \ldots, W(\bar{g}_{n+1}) \rangle$ as we needed to show. $\square$

5. Construction of a basis of *W*

We will proceed with the notations introduced in Sections 3 and 4. Suppose $B = (\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_{2k}) \in \Omega_{2k}$ with $k \geqslant 2$. Then for all $i \in \{1, \ldots, 2k\}$, we define

$$B(i) := (\bar{g}_1, \ldots, \bar{g}_{i-2}, \bar{g}_{i-1} + \bar{g}_{i+1}, \bar{g}_{i+2}, \ldots, \bar{g}_{2k}).$$

In particular,

$$B(1) = (\bar{g}_3, \bar{g}_4, \ldots, \bar{g}_{2k}),$$
$$B(2) = (\bar{g}_1 + \bar{g}_3, \bar{g}_4, \ldots, \bar{g}_{2k}),$$
$$B(2k-1) = (\bar{g}_1, \ldots, \bar{g}_{2k-3}, \bar{g}_{2k-2} + \bar{g}_{2k}),$$
$$B(2k) = (\bar{g}_1, \ldots, \bar{g}_{2k-2}).$$

Obviously, $B(i) \in \Omega_{2k-2}$ for all $i \in \{1, \ldots, 2k\}$. We call $\bar{g}_1, \ldots, \bar{g}_{i-2}, \bar{g}_{i-1} + \bar{g}_{i+1}, \bar{g}_{i+2}, \ldots, \bar{g}_{2k}$, respectively the first, the second, …, the $(2k-2)$th *component* of $B(i)$. Obviously, the following holds:

Property I. *If* $\bar{v}$ *is one of the components of* $B(i)$, *then* $(\bar{g}_i, \bar{v}) = 0$.

If $i_1, \ldots, i_l$ *are* $l \in \{1, \ldots, k-1\}$ *strictly positive integers such that* $i_j \leqslant 2k + 2 - 2j$ *for every* $j \in \{1, \ldots, l\}$, *then we define*

$$B(i_1, i_2, \dots, i_l) := B(i_1)(i_2)\cdots(i_l) \in \Omega_{2k-2l}.$$

Clearly, the following two properties hold:

Property II. *Let* $\bar{v} = \epsilon_1\bar{g}_1 + \epsilon_2\bar{g}_2 + \cdots + \epsilon_{2k}\bar{g}_{2k}$ *be the mth component of* $B(i_1, i_2, \dots, i_l)$, $1 \leqslant m \leqslant 2k - 2l$. *Then* (i) $\epsilon_1, \epsilon_2, \dots, \epsilon_{2k} \in \{0, 1\}$, (ii) $(\epsilon_1, \epsilon_3, \dots, \epsilon_{2k-1}) = (0, 0, \dots, 0)$ *if m is even, and* (iii) $(\epsilon_2, \epsilon_4, \dots, \epsilon_{2k}) = (0, 0, \dots, 0)$ *if m is odd.*

Property III. *Let* $\bar{v} = \epsilon_1\bar{g}_1 + \epsilon_2\bar{g}_2 + \cdots + \epsilon_{2k}\bar{g}_{2k}$ *and* $\bar{v}' = \epsilon'_1\bar{g}_1 + \epsilon'_2\bar{g}_2 + \cdots + \epsilon'_{2k}\bar{g}_{2k}$ *be the mth, respectively m′th component of* $B(i_1, i_2, \dots, i_l)$, $1 \leqslant m < m' \leqslant 2k - 2l$. *Let i, respectively i′, denote the largest, respectively smallest, element in* $\{1, 2, \dots, 2k\}$ *such that* $\epsilon_i \neq 0$, *respectively* $\epsilon'_{i'} \neq 0$. *Then* $i < i'$.

Before proceeding with the construction of a basis for W, we will need some definitions. Let $k_1, k_2 \in \mathbb{N} \setminus \{0\}$. With every $\bar{u}_1 \in \bigwedge^{k_1} V$ and every $\bar{u}_2 \in \bigwedge^{k_2} V$, there naturally corresponds an element $\bar{u}_1 \wedge \bar{u}_2$ of $\bigwedge^{k_1+k_2} V$. If $\bar{u}_1 \in \bigwedge^{k_1} V$ and if $(\bar{u}'_1, \bar{u}'_2, \dots, \bar{u}'_l)$ is an l-tuple ($l \geqslant 1$) of elements of $\bigwedge^{k_2} V$, then $\bar{u}_1 \wedge (\bar{u}'_1, \bar{u}'_2, \dots, \bar{u}'_l)$ denotes the l-tuple $(\bar{u}_1 \wedge \bar{u}'_1, \bar{u}_1 \wedge \bar{u}'_2, \dots, \bar{u}_1 \wedge \bar{u}'_l)$. If $(\bar{u}_1, \dots, \bar{u}_{l_1})$, respectively $(\bar{u}'_1, \dots, \bar{u}'_{l_2})$, is an l_1-tuple, respectively l_2-tuple, of elements of $\bigwedge^{k_1} V$ ($l_1, l_2 \geqslant 1$), then we denote by $(\bar{u}_1, \dots, \bar{u}_{l_1}) * (\bar{u}'_1, \dots, \bar{u}'_{l_2})$ the $(l_1 + l_2)$-tuple $(\bar{u}_1, \dots, \bar{u}_{l_1}, \bar{u}'_1, \dots, \bar{u}'_{l_2})$.

Again, let $k \in \mathbb{N} \setminus \{0\}$ and let $B = (\bar{g}_1, \bar{g}_2, \dots, \bar{g}_{2k}) \in \Omega_{2k}$. We will now define in a recursive way tuples $\Delta^k_i[B]$, $i \in \{1, 2, \dots, k + 1\}$, consisting of vectors of $\bigwedge^k V$.

- For $k = 1$, let $\Delta^1_1[B]$ and $\Delta^1_2[B]$ be the following 1-tuples:

$$\Delta^1_1[B] := (\bar{g}_1),$$
$$\Delta^1_2[B] := (\bar{g}_2).$$

- If $k \geqslant 2$, then we define

$$\Delta^k_i[B] := \bar{g}_i \wedge (\Delta^{k-1}_{i^*}[B(i)] * \Delta^{k-1}_{i^*+1}[B(i)] * \cdots * \Delta^{k-1}_k[B(i)]),$$

where $i^* := \max(i - 1, 1)$. We also define

$$\Delta^k[B] := \Delta^k_1[B] * \Delta^k_2[B] * \cdots * \Delta^k_{k+1}[B].$$

If $\Delta^k[B] = (\bar{u}_1, \bar{u}_2, \dots, \bar{u}_N)$ and $i, j \in \{1, 2, \dots, N\}$ with $i < j$, then we say that $\bar{u}_i$ *comes earlier* than $\bar{u}_j$ (in $\Delta^k[B]$).

By Property I, every component of $\Delta^k[B]$ can be written as $\bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_k$, where $(\bar{v}_1, \dots, \bar{v}_k)$ is a nice k-tuple. If the vectors $\bar{v}_1, \bar{v}_2, \dots, \bar{v}_k$ are chosen in the natural way, i.e. if they were obtained by literally applying the recursive definition, then $\bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_k$ is called the *natural representation* of the component of $\Delta^k[B]$. Now, take an arbitrary component of $\Delta^k[B]$ and let $\bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_k$ be its natural representation. For every $i \in \{1, \dots, k\}$, put $\bar{v}_i = \epsilon^{(i)}_1\bar{g}_1 + \epsilon^{(i)}_2\bar{g}_2 + \cdots + \epsilon^{(i)}_{2k}\bar{g}_{2k}$ with $\epsilon^{(i)}_j \in \{0, 1\}$ for all $j \in \{1, \dots, 2k\}$. If j is the largest value such that $\epsilon^{(i)}_j = 1$, then we define $\bar{w}_i := \bar{g}_j$. We call $\bar{w}_1 \wedge \bar{w}_2 \wedge \cdots \wedge \bar{w}_k$ the *largest content* of $\bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_k$.

Property IV. *Let $\bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_k$ be the natural representation of a component $\bar{u}$ of $\Delta^k[B]$ and let $\bar{w}_1 \wedge \bar{w}_2 \wedge \cdots \wedge \bar{w}_k$ denote its largest content. For every $j \in \{1, 2, \ldots, k\}$, let $i_j \in \{1, \ldots, 2k\}$ such that $\bar{w}_j = \bar{g}_{i_j}$. Then $i_1 < i_2 < \cdots < i_k$.*

Proof. Obviously, $\bar{u}$ is a component of $\Delta^k_{i_1}[B]$. Put $\bar{v}_2 = \sum_{j=1}^{2k} \epsilon_j \bar{g}_j$. Then i_2 is the largest $j \in \{1, \ldots, 2k\}$ for which $\epsilon_j \neq 0$. Since $\bar{v}_2 \wedge \cdots \wedge \bar{v}_k$ is a component of

$$\Delta^{k-1}_{i^*}[B(i_1)] * \Delta^{k-1}_{i^*+1}[B(i_1)] * \cdots * \Delta^{k-1}_{k}[B(i_1)], \quad i^* = \max(i_1 - 1, 1),$$

$\bar{v}_2$ is the jth component of $B(i_1)$ for some $j \geqslant i_1 - 1$. This implies that $i_2 > i_1$. The property now readily follows from an inductive argument taking into account Property III. □

Definition 1. (i) If $\bar{u}, i_1, i_2, \ldots, i_k$ are as in Property IV, then we define $I(\bar{u}) := (i_1, i_2, \ldots, i_k)$.

(ii) Let $i_1, i_2, \ldots, i_k, i'_1, i'_2, \ldots, i'_k$ be elements of $\{1, 2, \ldots, 2k\}$ such that $i_1 < i_2 < \cdots < i_k$ and $i'_1 < i'_2 < \cdots < i'_k$. Then we say that $(i_1, i_2, \ldots, i_k) < (i'_1, i'_2, \ldots, i'_k)$ if and only if $i_j < i'_j$ where j is the smallest element of $\{1, 2, \ldots, k\}$ for which $i_j \neq i'_j$.

Property V. *If $\bar{u}_1$ and $\bar{u}_2$ are components of $\Delta^k[B]$ such that $\bar{u}_1$ comes earlier than $\bar{u}_2$, then $I(\bar{u}_1) < I(\bar{u}_2)$.*

Proof. Let $\bar{u}_1$ be a component of $\Delta^k_{i_1}[B]$ and let $\bar{u}_2$ be a component of $\Delta^k_{i'_1}[B]$. Since $\bar{u}_1$ comes earlier than $\bar{u}_2$, $i_1 \leqslant i'_1$. If $i_1 < i'_1$, then obviously, $I(\bar{u}_1) < I(\bar{u}_2)$. If $i_1 = i'_1$, then $\bar{u}_1 = \bar{g}_{i_1} \wedge \bar{v}_1$ and $\bar{u}_2 = \bar{g}_{i_1} \wedge \bar{v}_2$, where $\bar{v}_1$ and $\bar{v}_2$ are components of

$$\Delta^{k-1}_{i^*}[B(i_1)] * \Delta^{k-1}_{i^*+1}[B(i_1)] * \cdots * \Delta^{k-1}_{k}[B(i_1)]$$

($i^* = \max(i_1 - 1, 1)$) such that $\bar{v}_1$ comes earlier than $\bar{v}_2$. The property now readily follows from an inductive argument taking into account Property III. □

Proposition 5.1. *$\Delta^k[B]$ is a collection of linearly independent vectors of $\bigwedge^k V$.*

Proof. A basis of $\bigwedge^k V$ is given by

$$\Theta = \{\bar{g}_{i_1} \wedge \bar{g}_{i_2} \wedge \cdots \wedge \bar{g}_{i_k} | 1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant 2k\}.$$

Now, put $\Delta^k[B] = (\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_N)$. We must show that for every $i \in \{2, \ldots, N\}$, $\bar{u}_i$ is not a linear combination of the vectors $\bar{u}_1, \ldots, \bar{u}_{i-1}$. Put $I(\bar{u}_i) = (i^*_1, i^*_2, \ldots, i^*_k)$. Since $I(\bar{u}_1), I(\bar{u}_2), \ldots, I(\bar{u}_{i-1}) < I(\bar{u}_i)$, none of the vectors $\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_{i-1}$ has a component in $\bar{g}_{i^*_1} \wedge \bar{g}_{i^*_2} \wedge \cdots \wedge \bar{g}_{i^*_k}$ in its expansion as a linear combination of elements of Θ. Since $\bar{u}_i$ has such a component, $\bar{u}_i \notin \langle \bar{u}_1, \ldots, \bar{u}_{i-1} \rangle$. This proves the proposition. □

Now, let $B^* = (\bar{g}^*_1, \bar{g}^*_2, \ldots, \bar{g}^*_{2n})$ be a given special basis of V. Then $\bigwedge^n V$ is a $\binom{2n}{n}$-dimensional vector space over $\mathbb{K}$ with basis $\{\bar{g}^*_{i_1} \wedge \bar{g}^*_{i_2} \wedge \cdots \wedge \bar{g}^*_{i_n} | 1 \leqslant i_1 < i_2 < \cdots < i_n \leqslant 2n\}$. For every $i \in \{1, \ldots, n+1\}$, we define

$$\Delta^n_{\leqslant i} := \Delta^n_1[B^*] * \Delta^n_2[B^*] * \cdots * \Delta^n_i[B^*],$$

and we denote $\Delta^n_{\leqslant n+1}$ also by Δ^n. For every $i \in \{1, \ldots, 2n\}$, we define

$U_i := W(\bar{g}_i^*)$.

Recall that by Proposition 4.4, $W = \langle U_1, U_2, \ldots, U_{n+1}\rangle$.

Proposition 5.2. *For every $j \in \{1, \ldots, n+1\}$, $\Delta^n_{\leqslant j}$ is a generating set of $\langle U_1, U_2, \ldots, U_j\rangle$. In particular, Δ^n is a generating set of $W = \langle U_1, U_2, \ldots, U_{n+1}\rangle$.*

Proof. We will prove the proposition by induction on n.

First, suppose $n = 1$. Then $\Delta^1_1[B^*] = (\bar{g}_1^*)$, $\Delta^1_2[B^*] = (\bar{g}_2^*)$, $U_1 = \langle \bar{g}_1^*\rangle$ and $U_2 = \langle \bar{g}_2^*\rangle$. Obviously, $\Delta^1_{\leqslant 1} = (\bar{g}_1^*)$ is a generating set of U_1 and $\Delta^1_{\leqslant 2} = (\bar{g}_1^*, \bar{g}_2^*)$ is a generating set of $\langle U_1, U_2\rangle$.

Suppose now that $n \geqslant 2$ and that the proposition holds for smaller values of n.

Let $j \in \{1, \ldots, n+1\}$. Then $B^*(j) = (\bar{g}_1^*, \ldots, \bar{g}_{j-2}^*, \bar{g}_{j-1}^* + \bar{g}_{j+1}^*, \bar{g}_{j+2}^*, \ldots, \bar{g}_{2n}^*)$ and the space $\langle \bar{g}_1^*, \ldots, \bar{g}_{j-2}^*, \bar{g}_{j-1}^* + \bar{g}_{j+1}^*, \bar{g}_{j+2}^*, \ldots, \bar{g}_{2n}^*\rangle$ is a subspace of $(\bar{g}_j^*)^\perp$ complementary to $\langle \bar{g}_j^*\rangle$. The subspace U_j is the subspace of W generated by all vectors of the form $\bar{g}_j^* \wedge \bar{v}_2 \wedge \bar{v}_3 \wedge \cdots \wedge \bar{v}_n$, where $(\bar{v}_2, \bar{v}_3, \ldots, \bar{v}_n)$ is some nice $(n-1)$-tuple of vectors contained in $\langle \bar{g}_1^*, \ldots, \bar{g}_{j-2}^*, \bar{g}_{j-1}^* + \bar{g}_{j+1}^*, \bar{g}_{j+2}^*, \ldots, \bar{g}_{2n}^*\rangle$. So, by the induction hypothesis,

$$\bar{g}_j^* \wedge (\Delta_1^{n-1}[B^*(j)] * \Delta_2^{n-1}[B^*(j)] * \cdots * \Delta_n^{n-1}[B^*(j)])$$

is a generating set of U_j. For every $i \in \{1, \ldots, 2n-2\}$, let $U_{j,i}$ denote the subspace of U_j generated by all vectors of the form $\bar{g}_j^* \wedge \bar{g}_{j,i}^* \wedge \bar{v}_3 \wedge \cdots \wedge \bar{v}_n$, where $\bar{g}_{j,i}^*$ is the ith component of $B^*(j)$ and $(\bar{v}_3, \ldots, \bar{v}_n)$ is a nice $(n-2)$-tuple of vectors contained in the subspace generated by the components of $B^*(j)(i)$. Obviously, for every $i \in \mathbb{N}$ such that $1 \leqslant i \leqslant j-2$, $U_{j,i} \subseteq U_i \cap U_j$.

We will now proceed by induction on j. Since

$$\Delta_1^n[B^*] = \bar{g}_1^* \wedge (\Delta_1^{n-1}[B^*(1)] * \Delta_2^{n-1}[B^*(1)] * \cdots * \Delta_n^{n-1}[B^*(1)])$$

is a generating set of U_1 and

$$\Delta_2^n[B^*] = \bar{g}_2^* \wedge (\Delta_1^{n-1}[B^*(2)] * \Delta_2^{n-1}[B^*(2)] * \cdots * \Delta_n^{n-1}[B^*(2)])$$

is a generating set of U_2, $\Delta^n_{\leqslant 1}$ is a generating set of U_1 and $\Delta^n_{\leqslant 2} = \Delta^n_1[B^*] * \Delta^n_2[B^*]$ is a generating set of $\langle U_1, U_2\rangle$.

Suppose now that $j \geqslant 3$ and that the claim holds for smaller values of j. By the induction hypothesis:

(i) $\Delta^n_{\leqslant j-1}$ is a generating set of $\langle U_1, \ldots, U_{j-1}\rangle$;

(ii) $\bar{g}_j^* \wedge (\Delta_1^{n-1}[B^*(j)] * \Delta_2^{n-1}[B^*(j)] * \cdots * \Delta_n^{n-1}[B^*(j)])$ is a generating set of U_j;

(iii) $\bar{g}_j^* \wedge (\Delta_1^{n-1}[B^*(j)] * \Delta_2^{n-1}[B^*(j)] * \cdots * \Delta_{j-2}^{n-1}[B^*(j)])$ is a generating set of $\langle U_{j,1}, \ldots, U_{j,j-2}\rangle$.

Since $\langle U_{j,1}, \ldots, U_{j,j-2}\rangle \subseteq \langle U_1, \ldots, U_{j-1}\rangle$,

$$\begin{aligned}&\Delta^n_{\leqslant j-1} * (\bar{g}_j^* \wedge (\Delta_{j-1}^{n-1}[B^*(j)] * \Delta_j^{n-1}[B^*(j)] * \cdots * \Delta_n^{n-1}[B^*(j)]))\\ &\quad = \Delta^n_{\leqslant j-1} * \Delta^n_j[B^*]\\ &\quad = \Delta^n_{\leqslant j}\end{aligned}$$

is a generating set of $\langle U_1, U_2, \ldots, U_j\rangle$. $\square$

By Propositions 5.1 and 5.2, we immediately have:

Corollary 5.3. *Δ^n is an ordered basis of W.*

6. Properties of the components of Δ^n

In Corollary 5.3 we have seen that Δ^n is an ordered basis of W. We will now take a closer look at the structure of the components of Δ^n.

Definition 2. (1) If $\bar{b}$ is a component of Δ^n with natural representation $\bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_n$, then by Property II, either $\bar{v}_i \in \langle \bar{g}_1^*, \bar{g}_3^*, \ldots, \bar{g}_{2n-1}^* \rangle$ or $\bar{v}_i \in \langle \bar{g}_2^*, \bar{g}_4^*, \ldots, \bar{g}_{2n}^* \rangle$ ($i \in \{1, \ldots, n\}$). We denote by $N(\bar{b})$ the number of $i \in \{1, \ldots, n\}$ for which $\bar{v}_i \in \langle \bar{g}_1^*, \bar{g}_3^*, \ldots, \bar{g}_{2n-1}^* \rangle$.

(2) For all $k, l \in \{0, \ldots, n\}$, let $h_n(k, l)$ denote the number of components $\bar{b}$ of $\Delta_{k+1}^n[B^*]$ for which $N(\bar{b}) = n - l$.

Proposition 6.1. *For every $n \geqslant 1$ and all $k, l \in \{0, \ldots, n\}$, $h_n(k, l) = f_n(k, l) = g_n(k, l)$.*

Proof. Suppose first that $n = 1$. Then $B^* = (\bar{g}_1^*, \bar{g}_2^*)$, $\Delta_1^1[B^*] = (\bar{g}_1^*)$ and $\Delta_2^1[B^*] = (\bar{g}_2^*)$. Hence, $h_1(0, 0) = 1$, $h_1(0, 1) = 0$, $h_1(1, 0) = 0$ and $h_1(1, 1) = 1$.

Suppose now that $n \geqslant 2$. Then $\Delta_{k+1}^n[B^*]$ is equal to

$$\bar{g}_{k+1}^* \wedge (\Delta_{i^*}^{n-1}[B^*(k+1)] * \Delta_{i^*+1}^{n-1}[B^*(k+1)] * \cdots * \Delta_n^{n-1}[B^*(k+1)]),$$

where $i^* = \max(k, 1)$. If k is even, then $\bar{g}_{k+1}^* \in \langle \bar{g}_1^*, \bar{g}_3^*, \ldots, \bar{g}_{2n-1}^* \rangle$. Hence, $h_n(k, n) = 0$. Moreover, if $l \neq n$, then we obtain

$$h_n(k, l) = \sum_{j=\max(k,1)}^{n} h_{n-1}(j-1, l).$$

If k is odd, then $\bar{g}_{k+1}^* \in \langle \bar{g}_2^*, \bar{g}_4^*, \ldots, \bar{g}_{2n}^* \rangle$. Hence, $h_n(k, 0) = 0$. Moreover, if $l \neq 0$, then we obtain

$$h_n(k, l) = \sum_{j=k}^{n} h_{n-1}(j-1, l-1).$$

By the discussion of Section 2, it follows that $h_n(k, l) = f_n(k, l) = g_n(k, l)$ for all $n \in \mathbb{N} \setminus \{0\}$ and all $k, l \in \{0, \ldots, n\}$. $\square$

Corollary 6.2. *The number of components $\bar{b}$ of Δ^n for which $N(\bar{b}) = n - l$, $l \in \{0, \ldots, n\}$, is equal to $\binom{n}{l}^2 - \binom{n}{l-1} \cdot \binom{n}{l+1}$.*

Proof. The number of components $\bar{b}$ of Δ^n for which $N(\bar{b}) = n - l$ is equal to $\sum_{k=0}^{n} h_n(k, l) = \sum_{k=0}^{n} f_n(k, l) = f_{n+1}(0, l) = g_{n+1}(0, l) = \binom{n}{l} \cdot \binom{n+1}{l} - \binom{n}{l-1} \cdot \binom{n+1}{l+1} = \binom{n}{l} \cdot \left[\binom{n+1}{l} - \binom{n}{l-1}\right] - \binom{n}{l-1} \cdot \left[\binom{n+1}{l+1} - \binom{n}{l}\right] = \binom{n}{l} \cdot \binom{n}{l} - \binom{n}{l-1} \cdot \binom{n}{l+1}$. $\square$

Corollary 6.3. $|\Delta^n| = \binom{2n}{n} - \binom{2n}{n-2}$.

Proof. By Corollary 6.2, $|\Delta^n| = \sum_{l=0}^{n} \binom{n}{l}^2 - \sum_{l=0}^{n} \binom{n}{l-1} \cdot \binom{n}{l+1}$.

(i) We show that $\sum_{l=0}^{n} \binom{n}{l}^2 = \binom{2n}{n}$. The coefficients of $x^l y^{n-l}$ and $x^{n-l} y^l$ in the expansion of $(x+y)^n$ are equal to $\binom{n}{l}$. Hence, the coefficient of $x^n y^n$ in the expansion of $(x+y)^n \cdot (x+y)^n = (x+y)^{2n}$ is equal to $\sum_{l=0}^{n} \binom{n}{l}^2$. On the other hand we know that this coefficient is equal to $\binom{2n}{n}$.

(ii) We show that $\sum_{l=0}^{n} \binom{n}{l-1} \cdot \binom{n}{l+1} = \binom{2n}{n-2}$. The coefficient of $x^{l-1} y^{n-l+1}$, respectively $x^{n-l-1} y^{l+1}$, in the expansion of $(x+y)^n$ is equal to $\binom{n}{l-1}$, respectively $\binom{n}{l+1}$. Hence, the coefficient of $x^{n-2} y^{n+2}$ in the expansion of $(x+y)^n \cdot (x+y)^n = (x+y)^{2n}$ is equal to $\sum_{l=0}^{n} \binom{n}{l-1} \cdot \binom{n}{l+1}$. On the other hand, we know that this coefficient is equal to $\binom{2n}{n-2}$. □

Corollary 6.3 says that the subspace W of $\bigwedge^n V$ has dimension $\binom{2n}{n} - \binom{2n}{n-2}$.

Definition 3. (1) Let T_i, $i \in \{0, \ldots, n\}$, denote the subspace of W generated by all components $\bar{b}$ of Δ^n for which $N(\bar{b}) \geqslant n - i$. By Corollary 6.2, $\dim(T_i) = \sum_{l=0}^{i} \left[\binom{n}{l}^2 - \binom{n}{l-1} \cdot \binom{n}{l+1} \right]$.

(2) For every $i \in \{0, \ldots, n\}$, let T_i' denote the subspace of W generated by all vectors $\bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_n$, where $(\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n)$ is a nice n-tuple such that $\dim(\langle \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n \rangle \cap \langle \bar{g}_1^*, \bar{g}_3^*, \ldots, \bar{g}_{2n-1}^* \rangle) \geqslant n - i$.

Proposition 6.4. *For every $i \in \{0, \ldots, n\}$, $T_i' = T_i$.*

Proof. Let $\bar{b}$ be a component of Δ^n such that $N(\bar{b}) \geqslant n - i$. If $\bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_n$ is the natural representation of $\bar{b}$, then there are at least $n - i$ vectors $\bar{v}_j$ belonging to $\langle \bar{g}_1^*, \bar{g}_3^*, \ldots, \bar{g}_{2n-1}^* \rangle$. It follows that $\dim(\langle \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n \rangle \cap \langle \bar{g}_1^*, \bar{g}_3^*, \ldots, \bar{g}_{2n-1}^* \rangle) \geqslant n - i$. This proves that $T_i \subseteq T_i'$.

Conversely, let $(\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n)$ be a nice n-tuple of vectors of V such that $\dim(\langle \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n \rangle \cap \langle \bar{g}_1^*, \bar{g}_3^*, \ldots, \bar{g}_{2n-1}^* \rangle) = j \geqslant n - i$. Then $\bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_n = \bar{v}_1' \wedge \bar{v}_2' \wedge \cdots \wedge \bar{v}_n'$, where $\bar{v}_1', \bar{v}_2', \ldots, \bar{v}_j' \in \langle \bar{g}_1^*, \bar{g}_3^*, \ldots, \bar{g}_{2n-1}^* \rangle$. Now, expand $\bar{v}_1' \wedge \bar{v}_2' \wedge \cdots \wedge \bar{v}_n' \in W$ as a linear combination of base elements of Δ^n. In this expansion, one only needs base elements $\bar{b}$ of Δ^n with $N(\bar{b}) \geqslant j$. Hence, $\bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_n \in T_i$.

It follows that $T_i = T_i'$. □

7. Proof of Theorem 1.1

Let $n \geqslant 2$, let $\mathbb{K}$ be a field and let $V = V(2n, \mathbb{K})$ be a $2n$-dimensional vector space over $\mathbb{K}$ equipped with a symplectic form $(\cdot, \cdot)$. Let $DW(2n-1, \mathbb{K})$ denote the corresponding dual polar space. If α is a maximal totally isotropic subspace of V and if $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n$ is a generating set of α, then let $\wedge^n(\alpha)$ denote the line $\langle \bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_n \rangle$ of $\bigwedge^n V$. The map $\wedge^n$ defines the Grassmann-embedding e of $DW(2n-1, \mathbb{K})$ into a subspace $\mathrm{PG}(W)$ of $\mathrm{PG}(\bigwedge^n V)$.

Let $\{x, y\}$ be a pair of opposite points of $DW(2n-1, \mathbb{K})$. Then we can choose an ordered basis $(\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_{2n})$ in $V(2n, \mathbb{K})$ such that:

(i) $x = \langle \bar{e}_1, \bar{e}_3, \ldots, \bar{e}_{2n-1} \rangle$;
(ii) $y = \langle \bar{e}_2, \bar{e}_4, \ldots, \bar{e}_{2n} \rangle$;
(iii) with respect to the ordered basis $(\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_{2n})$, the symplectic form is represented by the following matrix:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 0 \end{pmatrix}.$$

Now, put

$$\begin{aligned} \bar{g}_1^* &= \bar{e}_1, \\ \bar{g}_{2i}^* &= \bar{e}_{2i}, \quad i \in \{1, \ldots, n\}, \\ \bar{g}_{2i-1}^* &= \bar{e}_{2i-1} - \bar{e}_{2i-3}, \quad i \in \{2, \ldots, n\}. \end{aligned}$$

Then $(\bar{g}_1^*, \bar{g}_2^*, \ldots, \bar{g}_{2n}^*)$ is a special basis of the symplectic vector space V. Moreover, $\langle \bar{g}_1^*, \bar{g}_3^*, \ldots, \bar{g}_{2n-1}^* \rangle = \langle \bar{e}_1, \bar{e}_3, \ldots, \bar{e}_{2n-1} \rangle = x$ and $\langle \bar{g}_2^*, \bar{g}_4^*, \ldots, \bar{g}_{2n}^* \rangle = y$.

We can use the special basis $(\bar{g}_1^*, \bar{g}_2^*, \ldots, \bar{g}_{2n}^*)$ to construct an ordered basis Δ^n of W, see Section 5. Let W_i, $i \in \{0, \ldots, n\}$, denote the subspace of W generated by all components $\bar{b}$ of Δ^n for which $N(b) = n - i$. Then obviously,

$$W = W_0 \oplus W_1 \oplus \cdots \oplus W_n.$$

Notice also that if $\bar{b}$ is a component of Δ^n for which $N(\bar{b}) = n - i$, then $\langle \bar{b} \rangle = e(z)$ for a point $z \in \Delta_i(x) \cap \Delta_{n-i}(y)$ (consider the natural representation of $\bar{b}$). This implies that $\mathrm{PG}(W_i) \subseteq \langle e(\Delta_i(x) \cap \Delta_{n-i}(y)) \rangle$. Conversely, if $z \in \Delta_i(x) \cap \Delta_{n-i}(y)$, then there exist vectors $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_{n-i} \in \langle \bar{g}_1^*, \bar{g}_3^*, \ldots, \bar{g}_{2n-1}^* \rangle$ and $\bar{v}_{n-i+1}, \bar{v}_{n-i+2}, \ldots, \bar{v}_{2n} \in \langle \bar{g}_2^*, \bar{g}_4^*, \ldots, \bar{g}_{2n}^* \rangle$ such that $z = \langle \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n \rangle$. Writing $\bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_n$ as a linear combination of the base elements of Δ^n, we see that we only need base elements $\bar{b}$ with $N(\bar{b}) = n - i$. This proves that

$$\mathrm{PG}(W_i) = \langle e(\Delta_i(x) \cap \Delta_{n-i}(y)) \rangle.$$

By Corollary 6.2,

$$\dim(W_i) = \binom{n}{i}^2 - \binom{n}{i-1} \cdot \binom{n}{i+1}.$$

We have shown in Corollary 6.3 that this implies that

$$\dim(W) = \sum_{i=0}^{n} \dim(W_i) = \binom{2n}{n} - \binom{2n}{n-2}.$$

Now, let T_i and T_i' be as defined in Section 6. Then obviously,

$$T_i = W_0 \oplus W_1 \oplus \cdots \oplus W_i$$

and $\mathrm{PG}(T_i')$ is the subspace of $\mathrm{PG}(W)$ generated by all points $e(z)$, where z is a point of $DW(2n-1, \mathbb{K})$ at distance at most i from x. By Proposition 6.4, $\mathrm{PG}(T_i') = \mathrm{PG}(T_i) = \mathrm{PG}(W_0 \oplus W_1 \oplus \cdots \oplus W_i)$. So,

$$\langle e(\Delta_i^*(x)) \rangle = \mathrm{PG}(W_0 \oplus W_1 \oplus \cdots \oplus W_i).$$

By reasons of symmetry, we also have

$$\langle e(\Delta_i^*(y)) \rangle = \mathrm{PG}(W_{n-i} \oplus W_{n-i+1} \oplus \cdots \oplus W_n).$$

This proves Theorem 1.1. □

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